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Some infinite integrals with powers of logarithms and the complete Bell polynomials

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Abstract

Modern computing tools, such as Computer Algebra, often allow a straightforward evaluation of mathematical expressions, for example by using recurrence relations. However, results so obtained may hide structures, which in some cases are not immediately recognized. This is discussed for a definite integral that is related to the higher derivatives of the gamma function. Two other, similar integrals are also considered.

Keywords: Infinite integrals; Complete Bell polynomials; Differentiation; Recurrence; Gamma function

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1. Introduction

Modern computing tools, such as Computer Algebra, allow in many cases the evaluation of mathematical expressions, for example via recurrence relations, which it would be too tedious to carry out by hand. However, results obtained in this way may hide structures, which in some cases are not easy to recognize. This is perhaps tolerable if one is interested in special cases only, but it is unsatisfactory when mathematical understanding is required or desirable.

A typical example of such a problem is the evaluation, in [4], of the definite integral

$$R_m(a, v) = \int_0^\infty e^{-ax} x^{v-1} \ln^m x \, dx \quad (\operatorname{Re} a > 0, \operatorname{Re} v > 0, m \in \mathbb{N}_0) \tag{1}$$

by repeated differentiation of the gamma function $\Gamma(x)$:

$$R_m(a, v) = \left(\frac{\partial}{\partial v}\right)^m \int_0^\infty e^{-ax} x^{v-1} \, dx = \left(\frac{\partial}{\partial v}\right)^m a^{-v} \Gamma(v). \tag{2}$$

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In some integral tables, e.g. in [3, No. 4.352 1, 4.358 2, 3, 4], one can find $R_m(a, v)$ for small values of m:

$$\begin{aligned} R_1(a, v) &= a^{-v}\Gamma(v)[\psi(v) - \ln a], \\ R_2(a, v) &= a^{-v}\Gamma(v)\{[\psi(v) - \ln a]^2 + \zeta(2, v)\}, \\ R_3(a, v) &= a^{-v}\Gamma(v)\{[\psi(v) - \ln a]^3 + 3[\psi(v) - \ln a]\zeta(2, v) - 2\zeta(3, v)\}, \\ R_4(a, v) &= a^{-v}\Gamma(v)\{[\psi(v) - \ln a]^4 + 6[\psi(v) - \ln a]^2\zeta(2, v) \\ &\qquad - 8[\psi(v) - \ln a]\zeta(3, v) + 3\zeta^2(v) + 6\zeta(4, v)\}, \end{aligned} \tag{3}$$

where $\psi(x)$ is the logarithmic derivative of $\Gamma(x)$ and

$$\zeta(n, x) = \sum_{j=0}^{\infty} \frac{1}{(j + x)^n} \quad (n \geqslant 2) \tag{4}$$

is the generalized zeta function. This function is related to the derivatives of $\psi(x)$ by

$$\psi^{(k)}(x) = (-1)^{k+1}k!\zeta(k + 1, x) \quad (k \in \mathbb{N}). \tag{5}$$

Formulae (3) can be obtained by repeated differentiation, using $\Gamma'(v) = \Gamma(v)\psi(v)$ after each step.

For larger m, it is convenient to apply a lemma, which is easy to prove by induction, easy to program in a Computer Algebra Language, and which allows, by recurrence, the calculation of successive derivatives of $\exp g(x)$ when $g(x)$ is given.

Lemma 1. *Let $g(x)$ be m times differentiable. Then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \mathrm{e}^{g(x)} = \mathrm{e}^{g(x)} G_m(x)$$

with $G_0(x) = 1$, $G_1(x) = g'(x)$, and

$$G_k(x) = G'_{k-1}(x) + G_1(x)G_{k-1}(x) \quad (k = 2, 3, \ldots, m).$$

A straightforward application of this lemma to

$$R_m(a, v) = \left(\frac{\partial}{\partial v}\right)^m \mathrm{e}^{\ln \Gamma(v) - v \ln a} = \left(\frac{\partial}{\partial v}\right)^m \mathrm{e}^{g(a, v)} = a^{-v}\Gamma(v)G_m(a, v)$$

leads, using (5) and the abbreviation $\phi = \phi(a, v) = \psi(v) - \ln a$, to the following expressions for $G_m(a, v)$:

$$G_1 = \phi,$$

$$G_2 = \phi^2 + \eta_2,$$

$$\begin{aligned}
G_3 &= \phi^3 + 3\eta_2\phi + \eta_3,\\
G_4 &= \phi^4 + 6\eta_2\phi^2 + 4\eta_3\phi + \eta_4,\\
G_5 &= \phi^5 + 10\eta_2\phi^3 + 10\eta_3\phi^2 + 5\eta_4\phi + \eta_5,\\
G_6 &= \phi^6 + 15\eta_2\phi^4 + 20\eta_3\phi^3 + 15\eta_4\phi^2 + 6\eta_5\phi + \eta_6, \ldots
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
\eta_2 &= \zeta(2, v),\\
\eta_3 &= -2\zeta(3, v),\\
\eta_4 &= 3\{\zeta^2(2, v) + 2\zeta(4, v)\},\\
\eta_5 &= -4\{5\zeta(2, v)\zeta(3, v) + 6\zeta(5, v)\},\\
\eta_6 &= 5\{3\zeta^3(2, v) + 18\zeta(2, v)\zeta(4, v) + 8\zeta^2(3, v) + 24\zeta(6, v)\}, \ldots
\end{aligned} \tag{7}$$

The quantities $G_m(a, v)$ in (6) clearly have a structure. In particular, the binomial coefficients and the absence of a term with ϕ^{m-1} are striking. However, based only on the information given in (7), finding a rule for the coefficients of the products $\Pi_h\zeta(k_h, v)$ in η_k, which apparently extend over all partitions $\{k_h\}$ of k with $k_h \geqslant 2$, seems less obvious.

It is the purpose of this note to show how these structures can be explained for this and two other, similar integrals. They are closely related to the complete Bell polynomials, and some properties of these polynomials are presented in the next section.

2. The complete Bell polynomials

In Lemma 1 we have given a simple recurrence formula for the calculation of the mth derivative of $\exp g(x)$. It is, however, well known that this derivative can be expressed in terms of known quantities by [7, p. 35]

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \mathrm{e}^{g(x)} = \mathrm{e}^{g(x)} Y_m(g'(x), g''(x), \ldots, g^{(m)}(x)), \tag{8}$$

where $Y_m(x_1, x_2, \ldots, x_m)$ is the mth (exponential) complete Bell polynomial, defined for $n = m$ by [2, p. 134]

$$\exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = 1 + \sum_{n=1}^{\infty} Y_n(x_1, x_2, \ldots, x_n)\frac{t^n}{n!}. \tag{9}$$

A combinatorial representation of Y_n is given by (simplified from [8, p. 173])

$$\begin{aligned}
Y_n(x_1, \ldots, x_n) &= \sum_{\pi(n)} \frac{n!}{k_1! \cdots k_n!}\left(\frac{x_1}{1!}\right)^{k_1} \cdots \left(\frac{x_n}{n!}\right)^{k_n}\\
&= \sum_{\pi(n)} (n; k_1, \ldots, k_n)' x_1^{k_1} \cdots x_n^{k_n},
\end{aligned} \tag{10}$$

where

$$(n; k_1, \dots, k_n)' = \frac{n!}{(1!)^{k_1} k_1! \cdots (n!)^{k_n} k_n!}$$

are the multinomial coefficients (of the third kind) defined with this notation in [1, No. 24.1.2]. The sum has to be taken over all partitions $\pi(n)$ of n, i.e., over all sets of integers $\{k_h\}$ $(h = 1, 2, \dots, n)$ with

$$\sum_{h=1}^{n} h k_h = n \quad (0 \leqslant k_h \leqslant n). \tag{11}$$

The coefficients $(n; k_1, \dots, k_n)'$ for $n = 1, 2, \dots, 10$ can be found in [1, Table 24.2].

The complete Bell polynomials have integer coefficients and the first five are [2, p. 307]

$$Y_1(x_1) = x_1,$$

$$Y_2(x_1, x_2) = x_1^2 + x_2,$$

$$Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3,$$

$$Y_4(x_1, x_2, x_3, x_4) = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4,$$

$$Y_5(x_1, x_2, x_3, x_4, x_5) = x_1^5 + 10x_1^3x_2 + 10x_1^2x_3 + 15x_1x_2^2 + 5x_1x_4 + 10x_2x_3 + x_5.$$

It is not difficult to rewrite Y_n in a different form, as a polynomial in the variable x_1:

$$Y_n = \sum_{j=0}^{n} c_{n,j} y_j(x_2, \dots, x_j) x_1^{n-j}, \quad y_0, y_1 = \text{const.}$$

This leads to

$$Y_1 = x_1,$$

$$Y_2 = x_1^2 + y_2,$$

$$Y_3 = x_1^3 + 3y_2x_1 + y_3,$$

$$Y_4 = x_1^4 + 6y_2x_1^2 + 4y_3x_1 + y_4,$$

$$Y_5 = x_1^5 + 10y_2x_1^3 + 10y_3x_1^2 + 5y_4x_1 + y_5,$$

$$Y_6 = x_1^6 + 15y_2x_1^4 + 20y_3x_1^3 + 15y_4x_1^2 + 6y_5x_1 + y_6, \dots$$

where

$$y_2 = x_2,$$

$$y_3 = x_3,$$

$$y_4 = 3x_2^2 + x_4,$$

$$y_5 = 10x_2x_3 + x_5,$$

$$y_6 = 15x_2^3 + 15x_2x_4 + 10x_3^2 + x_6, \ldots$$

The structure of the Y_n represented in this way is explained by

Lemma 2. *The complete Bell polynomials can be represented by*

$$Y_n(x_1, \ldots, x_n) = x_1^n + \sum_{j=2}^{n} \binom{n}{j} y_j(x_2, \ldots, x_j) x_1^{n-j},$$

where

$$y_j(x_2, \ldots, x_j) = \sum_{\pi_0(j)} \frac{j!}{k_2! \cdots k_j!} \left(\frac{x_2}{2!}\right)^{k_2} \cdots \left(\frac{x_j}{j!}\right)^{k_j}$$
$$= \sum_{\pi_0(j)} (j; 0, k_2, \ldots, k_j)' x_2^{k_2} \cdots x_j^{k_j}.$$

This sum has to be taken over those partitions $\pi_0(j)$ of j for which the first element k_1 vanishes, i.e. over all $(j-1)$-tuples of integers $\{k_h\}$ $(h = 2, \ldots, j)$ with

$$\sum_{h=2}^{j} hk_h = j \quad (0 \leqslant k_h \leqslant j).$$

The polynomials y_j have integer coefficients and are homogeneous of degree j in the sense of the definition of j. Further, y_j does not depend on x_{j-1}.

Proof. It is easily seen from (10) that $c_{n,0} = y_0 = 1$ and $y_1 = 0$, since

$$1 \cdot k_1 + \sum_{h=2}^{n} hk_h = n$$

has only the solution $\{k_h\} = \{n, 0, \ldots, 0\}$ for $k_1 = n$ and no solution at all for $k_1 = n - 1$. Thus, we obtain from (10) by separating x_1, using the notation $k_1 = j$,

$$Y_n = x_1^n + \sum_{j=0}^{n-2} \frac{1}{j!} x_1^j \sum_{\pi_0(n-j)} \frac{n!}{k_2! \cdots k_{n-j}!} \left(\frac{x_2}{2!}\right)^{k_2} \cdots \left(\frac{x_{n-j}}{(n-j)!}\right)^{k_{n-j}},$$

where the second sum has to be taken over the partitions $\pi_0(n-j)$ of $n-j$ with $k_1 = 0$, i.e. over all $(n-j-1)$-tuples $\{k_h\}$ $(h = 2, \ldots, n-j)$ with

$$\sum_{h=2}^{n-j} hk_h = n - j \quad (0 \leqslant k_h \leqslant n - j),$$

where a set $\{k_h\}$ with $k_{n-j-1} \neq 0$ is not possible. By replacing j by $n-j$, we obtain the relation stated. The fact that Y_n has integer coefficients is clearly transferred to y_j. $\square$

3. The integrals

We now return to the integral (1) and prove (empty sums are to be replaced by zero, empty products by one)

Theorem 3. *Let*

$$G_m(a, v) = \frac{a^v}{\Gamma(v)} \int_0^\infty \mathrm{e}^{-ax} x^{v-1} \ln^m x \,\mathrm{d}x \quad (\operatorname{Re} a > 0, \operatorname{Re} v > 0, m \in \mathbb{N}_0)$$

and

$$\phi(a, v) = \frac{\partial}{\partial v} \ln a^{-v} \Gamma(v) = \psi(v) - \ln a.$$

Let further

$$\zeta(k, v) = \sum_{l=0}^{\infty} \frac{1}{(l+v)^k} \quad (k \geqslant 2)$$

be the generalized zeta function. $G_m(a, v)$ can then be expressed by

$$G_m(a, v) = \phi^m(a, v) + \sum_{j=2}^{m} \binom{m}{j} \eta_j(v) \phi^{m-j}(a, v),$$

where

$$\eta_j(v) = (-1)^j \sum_{\pi_0(j)} (j; 0, k_2, \ldots, k_j)^* \zeta^{k_2}(2, v) \cdots \zeta^{k_j}(j, v),$$

and where

$$(j; k_1, \ldots, k_j)^* = \frac{j!}{1^{k_1} k_1! \cdots j^{k_j} k_j!}$$

are the multinomial coefficients (of the second kind) defined with this notation in [1, No. 24.1.2]. *The sum in $\eta_j(v)$ has to be taken over all partitions $\pi_0(j)$ of j with*

$$\sum_{h=2}^{n} h k_h = j \quad (0 \leqslant k_h \leqslant j).$$

Proof. From (8), we obtain by applying Lemma 2, with $\phi^{(k)}(a, v) = \psi^{(k)}(v)$ $(k \geqslant 1)$ and using (5),

$$\begin{aligned}\eta_j(v) &= \eta_j(\phi'(a, v), \ldots, \phi^{(j-1)}(a, v)) = \eta_j(\psi'(v), \ldots, \psi^{(j-1)}(v)) \\ &= \eta_j(1!\zeta(2, v), -2!\zeta(3, v), \ldots, (-1)^j (j-1)!\zeta(j, v))\end{aligned}$$

$$= (-1)^j \sum_{\pi_0(j)} \frac{j!}{k_2! \cdots k_j!} \left(\frac{\zeta(2,v)}{2}\right)^{k_2} \cdots \left(\frac{\zeta(j,v)}{j}\right)^{k_j}$$

$$= (-1)^j \sum_{\pi_0(j)} (j; 0, k_2, \ldots, k_j)^* \zeta^{k_2}(2,v) \cdots \zeta^{k_j}(j,v). \qquad \square$$

The coefficients $(n; k_1, \ldots, k_n)^*$ for $n = 1, 2, \ldots, 10$ can be found in [1, Table 24.2]. Theorem 3 explains the relations presented in (6) and (7), which are at the same time examples for $G_m(a, v)$, $(m = 1, 2, \ldots, 6)$ and $\eta_j(v)$. In particular, for $a = 1$, this theorem gives a representation of $\Gamma^{(m)}(v)/\Gamma(v)$ in terms of $\psi(v)$ and $\zeta(k, v)$.

We now consider the integral

$$M_m(\beta, v, \lambda) = \int_0^\infty \frac{x^{v-1} \ln^m x}{(1+\beta x)^\lambda} \, dx = \frac{1}{\Gamma(\lambda)} \left(\frac{\partial}{\partial v}\right)^m \beta^{-v} \Gamma(v) \Gamma(\lambda - v),$$

which has been discussed in [5], and for which results calculated by using Computer Algebra for general values of β, v, λ have been given. The structure of these results is explained by

Theorem 4. *Let*

$$G_m(\beta, v, \lambda) = \frac{\beta^v \Gamma(\lambda)}{\Gamma(v)\Gamma(\lambda - v)} \int_0^\infty \frac{x^{v-1} \ln^m x}{(1+\beta x)^\lambda} \, dx \quad (|\arg \beta| < \pi, 0 < \operatorname{Re} v < \operatorname{Re} \lambda, m \in \mathbb{N}_0)$$

and

$$\phi(\beta, v, \lambda) = \frac{\partial}{\partial v} \ln \beta^{-v} \Gamma(v) \Gamma(\lambda - v) = \psi(v) - \psi(\lambda - v) - \ln \beta.$$

Let further $\zeta(k, v)$ and $(j; k_1, \ldots, k_j)^*$ be as defined in Theorem 3. $G_m(\beta, v, \lambda)$ can then be expressed by

$$G_m(\beta, v, \lambda) = \phi^m(\beta, v, \lambda) + \sum_{j=2}^m \binom{m}{j} \eta_j(v, \lambda) \phi^{m-j}(\beta, v, \lambda),$$

where

$$\eta_j(v, \lambda) = (-1)^j \sum_{\pi_0(j)} (j; 0, k_2, \ldots, k_j)^* \varphi^{k_2}(2, v, \lambda) \cdots \varphi^{k_j}(j, v, \lambda),$$

$$\varphi(i, v, \lambda) = \zeta(i, v) + (-1)^i \zeta(i, \lambda - v)$$

and where, as in Theorem 3, the sum in $\eta_j(v, \lambda)$ has to be taken over all partitions $\pi_0(j)$.

Proof. With $\phi(\beta, v, \lambda)$ instead of $\phi(a, v)$ and

$$\phi^{(k)}(\beta, v, \lambda) = \left(\frac{\partial}{\partial v}\right)^k \phi(\beta, v, \lambda) = \psi^{(k)}(v) - (-1)^k \psi^{(k)}(\lambda - v)$$

$$= (-1)^{k+1} k! \varphi(k+1, v, \lambda) \quad (k \geqslant 1)$$

instead of $\phi^{(k)}(a, v)$, the proof follows the same line as for Theorem 3. $\square$

At last, we consider the integral

$$N_m(\nu, \lambda) = \int_0^1 \frac{x^{\nu-1} \ln^m x}{(1-x)^\lambda} \, dx = \Gamma(1-\lambda) \left(\frac{\partial}{\partial \nu}\right)^m \frac{\Gamma(\nu)}{\Gamma(1+\nu-\lambda)}, \tag{12}$$

which has been discussed in [6]. For this integral, we have to distinguish between the cases $\lambda \neq l$ and $\lambda = l$ $(l \in \mathbb{N})$. The structure of the results calculated in [6] by using Computer Algebra for general values of ν and $\lambda \neq l$ is explained by

Theorem 5. *Let*

$$G_m(\nu, \lambda) = \frac{\Gamma(1+\nu-\lambda)}{\Gamma(\nu)\Gamma(1-\lambda)} \int_0^1 \frac{x^{\nu-1} \ln^m x}{(1-x)^\lambda} \, dx \quad (\operatorname{Re} \nu > 0,\ \operatorname{Re} \lambda < m+1,\ m \in \mathbb{N}_0)$$

and

$$\phi(\nu, \lambda) = \frac{\partial}{\partial \nu} \ln \frac{\Gamma(\nu)}{\Gamma(1+\nu-\lambda)} = \psi(\nu) - \psi(1+\nu-\lambda).$$

Let further $\zeta(k, \nu)$ and $(j; k_1, \ldots, k_j)^*$ be as defined in Theorem 3. $G_m(\nu, \lambda)$ can then be expressed by

$$G_m(\nu, \lambda) = \phi^m(\nu, \lambda) + \sum_{j=2}^m \binom{m}{j} \eta_j(\nu, \lambda) \phi^{m-j}(\nu, \lambda),$$

where

$$\eta_j(\nu, \lambda) = (-1)^j \sum_{\pi_0(j)} (j; 0, k_2, \ldots, k_j)^* \varphi^{k_2}(2, \nu, \lambda) \cdots \varphi^{k_j}(j, \nu, \lambda),$$

$$\varphi(i, \nu, \lambda) = \zeta(i, \nu) - \zeta(i, 1+\nu-\lambda)$$

and where, as in Theorem 3, the sum in $\eta_j(\nu, \lambda)$ has to be taken over all partitions $\pi_0(j)$.

Proof. With $\phi(\nu, \lambda)$ instead of $\phi(a, \nu)$ and

$$\phi^{(k)}(\nu, \lambda) = \left(\frac{\partial}{\partial \nu}\right)^k \phi(\nu, \lambda) = \psi^{(k)}(\nu) - \psi^{(k)}(1+\nu-\lambda)$$
$$= (-1)^{k+1} k! \varphi(k+1, \nu, \lambda) \quad (k \geqslant 1)$$

instead of $\phi^{(k)}(a, \nu)$, the proof follows the same line as for Theorem 3. □

For $\lambda = l$ $(l \in \mathbb{N})$, special cases of the integral (12), computed with the help of Computer Algebra, have been given for $m = 1, \ldots, 4$ in [6]. The structure of these results for general values of ν, l and m is explained by

Theorem 6. *Let*

$$N_m(\nu, l) = \int_0^1 \frac{x^{\nu-1}\ln^m x}{(1-x)^l}\,\mathrm{d}x \quad (\operatorname{Re}\nu > 0, 0 < l \leqslant m, m \in \mathbb{N}),$$

and let

$$V_p(\xi_1, \xi_2, \ldots, \xi_p) = Y_p(\xi_1, -1!\,\xi_2, \ldots, (-1)^{p-1}(\mathrm{p}-1)!\,\xi_p), \quad V_0 = 1,$$

where $Y_p(x_1, \ldots, x_p)$ is the pth complete Bell polynomial; let further

$$s(\nu, l, k) = \sum_{j=1}^{l-1} \frac{1}{(\nu - j)^k}$$

and $\zeta(k, \nu)$ be the generalized zeta function. $N_m(\nu, l)$ can then be expressed by

$$N_m(\nu, l) = \frac{(-1)^l m!}{(l-1)!} \prod_{j=1}^{l-1} (\nu - j)$$
$$\times \sum_{k=0}^{m} \frac{(-1)^{k-1}}{(m-k)!} \tilde{\zeta}(k+1, \nu)\, V_{m-k}(s(\nu, l, 1), s(\nu, l, 2), \ldots, s(\nu, l, m-k)),$$

where $\tilde{\zeta}(1, \nu) = -\psi(\nu)$ and $\tilde{\zeta}(k+1, \nu) = \zeta(k+1, \nu)$ for $k \geqslant 1$.

Proof. It has been shown in [6] that $N_m(\nu, l)$ can be written as

$$N_m(\nu, l) = \frac{(-1)^l}{(l-1)!} \left(\frac{\partial}{\partial \nu}\right)^m \psi(\nu) \prod_{j=1}^{l-1} (\nu - j).$$

By using the Leibniz formula, we obtain

$$N_m(\nu, l) = \frac{(-1)^l}{(l-1)!} \sum_{k=0}^{m} \binom{m}{k} \psi^{(k)}(\nu) \left[\exp \sum_{j=1}^{l-1} \ln(\nu - j)\right]^{(m-k)}.$$

Carrying out the differentiations according to (8) and using (5) leads to the result stated. □

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